Diophantine conditions and approximations of irrationals by rationals

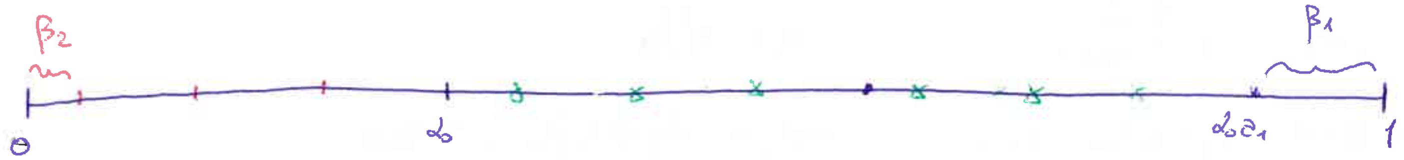
In Pfeiffer's result, α was very well approximated by rationals, and consequently $|\lambda^n - 1|$ admit very small values, which allow the coefficients ~~of~~ ϕ_n of the conjugacy to grow very fast and not allow convergence.

The natural strategy to find $\alpha \in \mathcal{L}$ consists in finding irrational numbers badly approximated by rationals.

Irrational rotations and continued fractions.

Consider the multiplication by $\lambda = e^{2\pi i \alpha}$ ($\alpha \in \mathbb{R}$) on S^1. It is conjugated to the translation by α on $\mathbb{R}/\mathbb{Z}$.

Up to replacing α by $\alpha_0 = \{\alpha\}$, we may think $\alpha_0 \in [0,1)$. (write $\alpha = \underset{\lfloor\alpha\rfloor}{\underset{\shortparallel}{a_0}} + \alpha_0$).



We look for the first ~~...~~ moment a_1 when $\alpha_0 \cdot a_1$ is closer to 1 than α_0:

Hence $a_1 = \left\lfloor \frac{1}{\alpha_0} \right\rfloor$. we get that $1 - \alpha_0 a_1 =: x_1 < 0$, and $\beta_1 = |x_1| < \alpha_0$.

Hence $\beta_1 = \alpha_0 \alpha_1$, for some $\alpha_1 \in [0,1)$. Notice that $\alpha_1 \neq 0$ as far as $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Notice that, to get to $\alpha_0 a_1$, we did $q_1 = a_1$ iterates, and we did almost $p_1 = a_0 a_1 + 1$ turns. In other terms: $q_1 \alpha - p_1 = x_1$.)

- q_1 is the first moment so that $\|q_1 \alpha\| < x_0 = \alpha_0$, where $\|\cdot\| = \mathrm{dist}(\cdot, \mathbb{Z})$, and the integer realizing the distance is p_1.

We repeat the process: By taking a_2 times a_1 iterates, we are translating by $a_2 x_1 = a_2 \beta_1$.

We look for a_2 so that we get $a_2 \beta_1$ close to α_0.

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Hence $a_2 = \left\lfloor \frac{\alpha_0}{\beta_1} \right\rfloor = \left\lfloor \frac{1}{\alpha_1} \right\rfloor$, and denote $\alpha_2 = \left\{ \frac{1}{\alpha_1} \right\}$, and $\beta_2 = \alpha_2 \beta_1 = \alpha_2 \alpha_1 \alpha_0$.

~~We obtained~~ by taking the $(\underbrace{1 + q_1 a_2}_{\shortparallel \atop q_2})$-th iterate, we get a point x_2 at distance $\beta_2 < \beta_1$ from 0. In this case, we did $p_2 = a_2 p_1 + p_0$ turns.

Notice that for any $q \in \{1, .., q_2 - 1\}$, the points $\lfloor q\alpha \rfloor$ belong to the interval $(x_2 ; 1 + x_1] = (\beta_2 ; 1 - \beta_1]$. In particular q_2 is the first iterate so that $\|q\alpha\| < \beta_1$.

↑The reason is that these points $\lfloor q\alpha \rfloor$ are obtained as the translates by $-\beta_1$, starting from the previous orbits $\{\alpha_0 .., q_1 \alpha_0\}$.

By iterating this procedure, we get:

$$a_0 = \lfloor \alpha \rfloor, \quad a_k = \left\lfloor \frac{1}{\alpha_{k-1}} \right\rfloor$$

$$\alpha_0 = \{\alpha\}; \quad \alpha_k = \left\{ \frac{1}{\alpha_{k-1}} \right\}$$

$$p_{-1} = 1 \quad p_0 = a_0 \quad p_k = a_k p_{k-1} + p_{k-2}$$

$$q_{-1} = 0 \quad q_0 = 1 \quad q_k = a_k q_{k-1} + q_{k-2}$$

$$x_k = q_k \alpha - p_k ; \quad \beta_k = \alpha_0 \cdots \alpha_k$$

$$x_k = (-1)^k \beta_k$$

$$\inf\{ q \mid \|q\alpha\| < \beta_k \} = q_{k+1} ,$$

One can reinterpret this recursion as:

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = L_{a_0} \cdots L_{a_k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where} \quad L_a = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$$

(by recursion, $L_{a_0} = \begin{pmatrix} p_0 & p_{-1} \\ q_0 & q_{-1} \end{pmatrix}$, and if $L_{a_0} \cdots L_{a_{k-1}} \cancel{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix}$,

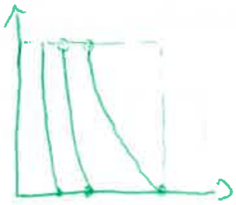
then $L_{a_0} \cdots L_{a_k} = \begin{pmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{pmatrix} \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & p_{k-1} \\ q_k & q_{k-1} \end{pmatrix}$.

L_a represents the map $M_a : a + \frac{1}{x}$ on $\mathbb{P}^1_{\mathbb{R}}$.

It follows that $\frac{p_k}{q_k} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_k}}} \overset{\text{notation}}{=} [a_0, a_1, \dots, a_k]$.

We denote: $\alpha = [a_0, a_1, a_2, \dots]$

Rem: The map G defined as $G(\alpha)=\left\{\frac{1}{\alpha}\right\}$ is sometimes called Gauss map. Its iteration is well defined on $(0,1)\setminus\mathbb{Q}$.



Other properties: ($\alpha\in\mathbb{R}\setminus\mathbb{Q}$)

- Let F_k be the Fibonacci sequence: $F_0=0$, $F_1=1$, $F_k=F_{k-1}+F_{k-2}$

$(0,1,1,2,3,5,8,\dots)$. Then $F_k=\frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^k-\left(\frac{1-\sqrt5}{2}\right)^k\right]\asymp\left(\frac{1+\sqrt5}{2}\right)^k$.

If $a_k=1\ \forall k$, then $q_k=F_{k+1}$.

In general, $q_k\geq F_{k+1}$, and q_k grows at least exponentially fast.

- $\alpha=\dfrac{p_k+p_{k-1}\alpha_k}{q_k+q_{k-1}\alpha_k}$; $\alpha-\dfrac{p_k}{q_k}=(-1)^k\dfrac{\beta_k}{q_k}$; $q_k\beta_{k+1}+q_{k+1}\beta_k=1$, $\beta_k\searrow 0$

Proof:

$\alpha=a_0+\alpha_0=M_{a_0}\left(\frac{1}{\alpha_0}\right)$ $\frac{1}{\alpha_0}=a_1+\alpha_1=M_{a_1}\left(\frac{1}{\alpha_1}\right)\Rightarrow\alpha=M_{a_0}\cdots M_{a_k}\left(\frac{1}{\alpha_k}\right)$.

We already saw the second property, then we have:

$$q_{k+1}\overset{q_k\alpha-p_k}{x_k}-q_kx_{k+1}=-q_{k+1}p_k+q_kp_{k+1}=(-1)^{k+2}\quad\leftarrow(\text{since }\det L_a=-1\ \forall a)$$

$(-1)^k(q_{k+1}\beta_k+q_k\beta_{k+1})$ which gives the third.

Since $\beta_k\searrow$, it tends to some $\beta_\infty\geq 0$. Then $1=q_{k+1}\beta_k+q_k\beta_{k+1}\ \forall k$

If $\beta_\infty>0$, then $1=2\beta_\infty\lim_{k\to\infty}q_k\to+\infty$: contradiction.

In particular, what we get is that $\forall\varepsilon>0$ (error), let k be the largest so that $\beta_{k-1}\geq\varepsilon$

Then $\|q\alpha\|\geq\varepsilon\ \forall q\in\mathbb{N}^*$, $q<q_k$.

In other terms: $\forall q<q_k$, we have $\left|\alpha-\frac{p}{q}\right|\geq\frac{\varepsilon}{q}$.

A similar construction consists in taking:

$a_0^- = \lceil \alpha \rceil$, $\alpha_0^- = \lceil \alpha \rceil - \alpha =: \{\alpha\}^- \in [0,1)$, and by recursion:

$\frac{1}{\alpha_{k-1}} = a_k^- + \alpha_k^-$, $a_k^- = \left\lceil \frac{1}{\alpha_{k-1}} \right\rceil$, $\alpha_k^- = \left\{ \frac{1}{\alpha_{k-1}} \right\}^-$ $\quad a_k \in \mathbb{N} \setminus \{0,1\}$ $\forall k \geq 1$

It gives: $\alpha = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \dots}} =: [a_0, a_1, a_2, \dots]^-$

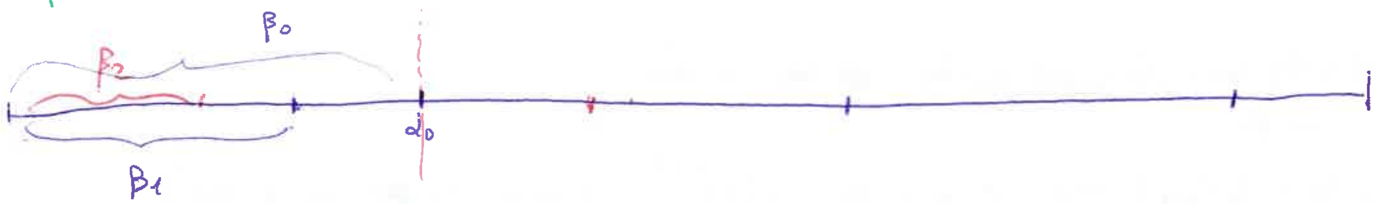
The theory is analogous with $L_a^- = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ (whose det. is +1)

$L_{a_0}^- \cdots L_{a_k}^- = \begin{pmatrix} p_k & -p_{k-1} \\ q_k & -q_{k-1} \end{pmatrix}$ with $p_k = a_k p_{k-1} - p_{k-2}$ $\quad p_{-1} = -1$ $\quad p_0 = a_0$

$q_k = a_k q_{k-1} - q_{k-2}$ $\quad q_{-1} = 0$ $\quad q_0 = 1$.

$M_a(\alpha) = a - \frac{1}{\alpha}$.

This operation corresponds to taking the multiples $q\alpha$ close to $\mathbb{Z}$ only on the positive side:



One can notice that in this case, if $\alpha \notin \mathbb{Q}$, then $a_k \geq 2$ and $\forall k\ \exists n > k$, $a_n \geq 3$.

In this case one can still show $q_n \to \infty$, even if much slower in general.

Diophantine conditions.

We consider the following sets of irrational numbers:

- $Q_2 = \{\text{quadratic irrationals}\}$ (zeros of polynomials of degree 2 with $\mathbb{Z}$-coeffs)
- $D(h)$, $h \geq 2$, the set of diophantine numbers of exponent h: for which: $\exists\, C$ constant so that $\forall p \in \mathbb{Z}, q \in \mathbb{N}^*$; $\left|\alpha - \frac{p}{q}\right| > \frac{C}{q^h}$.
- $\mathcal{D} = \bigcup_{h \geq 2} \mathcal{D}(h)$ set of diophantine numbers.
- B the set of Brjuno numbers, for which $\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty$.

Prop: let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha = [a_0, a_1, a_2, \dots]$. Then:

a. $Q_2 = \{\alpha \mid \text{the sequence } (a_n)_n \text{ is eventually periodic}\}$.

b. $\mathcal{D}(2) = \{\alpha \mid (a_n) \text{ is bounded}\}$ $\left(\{\text{Alg. numbers}\} \subset \mathcal{D}(2^+) = \bigcap_{h>2} \mathcal{D}(h)\right.$.

c. If $h \leq h' \Rightarrow D(h) \subset D(h')$

d. $D(2)$ has measure 0, but $\forall h > 2$, $D(h)$ has full measure.

e. $\mathcal{D} \subset B$.

Proof: a) ($\supseteq$) If (a_n) is eventually periodic, then $\exists\, n > m \geq 0$ so that $\alpha_n = \alpha_m$.

Then $\alpha_n = \dfrac{q_n \alpha - p_n}{q_{n-1}\alpha - p_{n-1}} = \dfrac{q_m \alpha - p_m}{q_{m-1}\alpha - p_{m-1}} = \alpha_m \Rightarrow \alpha \in Q_2$.

($\subseteq$) Idea: if α_0 satisfies $P_0(\alpha_0) = 0$, $P_0 = A_0 x^2 + B_0 x + C_0$, $A_0, B_0, C_0 \in \mathbb{Z}$, then α_k satisfies $A_k x^2 + B_k x + C_k = 0$. The set $\{A_k, B_k, C_k\}$ bounded

$\Rightarrow$ have only finitely many polynomials, hence zeros, and $\exists\, n > m \geq 0$ s.t. $a_n = a_m$. $\Rightarrow (a_k)$ is eventually periodic

(b) ② Suppose α is of bounded type: $a_n < A \quad \forall n \geq 1$.

$\Rightarrow p_{n+1} < (A+1)\, p_n \qquad (p_{n+1} = a_{n+1} p_n + p_{n-1} < A p_n + p_n)$, and $q_{n+1} < (A+1) q_n$.

Hence $\left| \alpha - \frac{p_n}{q_n} \right| = \frac{\beta_n}{q_n} = \frac{\beta_n q_{n+1}}{q_n q_{n+1}} \underset{\uparrow}{>} \frac{1}{2 q_n q_{n+1}} > \frac{1}{2(A+1) q_n^2} \Rightarrow \alpha \in D(2)$.

because $\beta_n q_{n+1} + \beta_{n+1} q_n = 1$, and $\beta_n \searrow$, $q_n \nearrow$.

⊆

Conversely: $\frac{c}{q_n^2} < \left| \alpha - \frac{p_n}{q_n} \right| = \frac{\beta_n q_{n+1}}{q_n q_{n+1}} < \frac{1}{q_n q_{n+1}}$.

$\rightsquigarrow a_{n+1} q_n < q_{n-1} + a_{n+1} q_n = q_{n+1} < \frac{q_n}{c} \Rightarrow a_{n+1} < \frac{1}{c} \quad \forall n$. OK

(c) is obvious by definition.

(d): We prove that $\mathrm{leb}(D(h) \cap [0,1]) = 1 \quad \forall h > 2$

Set $D_c(h) = \{ \alpha \in [0,1] \mid \left| \alpha - \frac{p}{q} \right| > \frac{c}{q^h} \}$. It is the complement of $\bigcup \left[\frac{p}{q} - \frac{c}{q^h}, \frac{p}{q} + \frac{c}{q^h} \right]$

where the union is when $\frac{p}{q}$ varies on $\mathbb{Q} \cap [0,1]$.

For a fixed q, there are (at most) q intervals, and the total length is given by

$\leq \sum_{q=1}^{\infty} \frac{2c}{q^h} \cdot q =: L_c$. Since $h > 2$, $h-1 > 1$, and: $L_c = 2c \cdot K_{<\infty}$.

~~[illegible]~~ But $D(h) = $ is the complement of the intersection of such union of intervals when $c \to 0$. It follows that $\mathrm{leb}(D(h)) = 1$.

(e) We argue as before, and if $\alpha \in D(h)$, then $\exists C$ constant s.t.

$\frac{C}{q_n^h} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} \Rightarrow q_{n+1} < \frac{q_n^{h-1}}{C}$

$\Rightarrow \frac{\log q_{n+1}}{q_n} < (h-1) \underline{\frac{\log q_n}{q_n} - \frac{\log C}{q_n}}$

defines a convergent series since q_n diverges at least exponentially fast $\square$

Alternative characterization of $\mathcal{B}$:

Prop: $\exists C$ constant so that $\forall k \geq 0$,

$$\left| \sum_{n=0}^{k} \beta_{n-1} \log \frac{1}{\alpha_n} - \sum_{n=0}^{k} \frac{\log q_{n+1}}{q_n} \right| < C.$$

In particular $\alpha \in \mathcal{B} \Leftrightarrow \underbrace{\sum_{n=0}^{\infty} \beta_{n-1} \log \frac{1}{\alpha_n}}_{=: B(\alpha)} < +\infty.$

Idea of the proof:

Step 1: $\exists C_1$ so that $\left| \sum_{n=0}^{k} \beta_{n-1} \log \frac{1}{\beta_n} - \sum_{n=0}^{\infty} \beta_{n-1} \log \frac{1}{\alpha_n} \right| < C_1$.

$$\left| \sum \beta_{n-1} \log \frac{1}{\beta_{n-1}} \right|, \text{ and } q_n \beta_{n-1} \leq 1$$

$\Rightarrow \frac{1}{\beta_{n-1}} \geq q_n$ $\nearrow$ exp fast.

Step 2: $\exists C_2$: $\left| \sum \frac{\log q_{n+1}}{q_n} - \sum \beta_{n-1} \log \frac{1}{\beta_n} \right| < C_2$.

$\square$

~~Theorem (Brjuno): $\mathcal{L} = \mathcal{B}$: i.e., the set of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that all f with multiplier $e^{2\pi i \alpha}$ is linearizable is exactly the set of Brjuno numbers.~~

Rem: coming back to the values $|\lambda^q - 1|$ $\lambda = e^{2\pi i \alpha}$. Then.

$|\lambda^q - 1| = |e^{2\pi i \alpha q} - 1| \sim 2\pi \|\alpha q\|$ as far as $\lambda^q \sim 1$.

We know that the best approximations are given by the convergents $\frac{p_k}{q_k}$:

$\|\alpha q_k\| = \beta_k$. Moreover: . $\|\alpha q\| \geq \beta_k$ $(> \beta_{k+1})$ $\forall q < q_{k+1}$.

From $q_k \beta_{k+1} + q_{k+1} \beta_k = 1$ we get $\frac{1}{2 q_{k+1}} < \beta_k < \frac{1}{q_{k+1}}$

Hence $\exists \underset{=}{\overset{2}{C_1}} < \underset{=}{\overset{2\pi}{C_2}}$ so that $|\lambda^q - 1| \geq |\lambda^{q_k} - 1|$ $\forall q,\ 0 < q < q_{k+1}$, with.

$$\frac{C_1}{q_{k+1}} \leq |\lambda^{q_k} - 1| \leq \frac{C_2}{q_{k+1}}.$$